Inverse Scattering problem and generalized optical theorem

@ ICNT workshop, MSU, 28 May 2015

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In short, I explain
1. condition for half-on-shell T-matrix
2. solution to inverse scattering in 3D space

1
7. t(k) of Gau
8.

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Inverse scattering problem and generalized optical theorem

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(Received 24 March 2014; accepted 22 January 2015; published online 5 February 2015)

1. Introduction --- history of inverse problems

- J.W.S.Rayleigh, 1877

  Given eigenfrequencies of a string,
  can one determine the density distribution of the string ?
  The theory of sound, Dover, 1945

- Marc Kac, 1966

  CAN ONE HEAR THE SHAPE OF A DRUM?

  MARK KAC, The Rockefeller University, New York

  To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

  "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

- Inverse scattering in 1D (~1960's)

$$\delta(k) \longleftrightarrow V(r)$$

← one-to-one

- Inverse scattering in 3D (2015, present work)

$$\langle \hat{\boldsymbol{k}}' | t(k) | \hat{\boldsymbol{k}} \rangle \longleftrightarrow \langle \boldsymbol{r}' | V | \boldsymbol{r} \rangle, \quad \langle \boldsymbol{r}' | V' | \boldsymbol{r} \rangle$$

← one-to-many

2. Current theory of inverse scattering (Marchenko)

$$\left\{-\nabla^2 + V(r)\right\}\psi_{\boldsymbol{k}}(\boldsymbol{r}) = k^2\psi_{\boldsymbol{k}}(\boldsymbol{r}) \Rightarrow \left\{\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - V(r) + k^2\right\}\psi_{k,l}(r) = 0$$

- **input**

  $\delta(k)$: phase shift

  ⬇

- **source term**

  $$A_0(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[S(k)-1]e^{ikt}dk$$

  ⬇

- **Marchenko eq.**

  $$A(r,t) = A_0(r+t) + \int_r^{\infty} A(r,s)A_0(s+t)ds$$

  ⬇

- **potential**

  $$V(r) = -2\frac{dA(r,r)}{dr}$$

one-to-one

$$\delta(k) \longleftrightarrow V(r)$$

function of a single variable

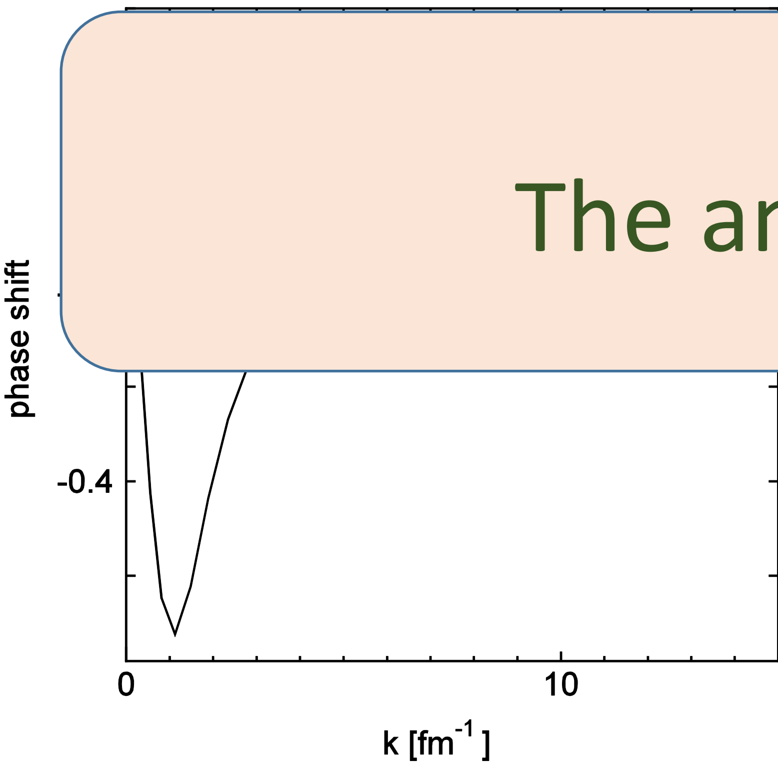
Textbook:

R.G.Newton, Scattering Theory of Waves and Particles, 2nd edition, Springer,1982

K.Chadan and P.C.Sabatier, Inverse Problems in Quantum Scattering Theory, 2nd edition, Springer,1989

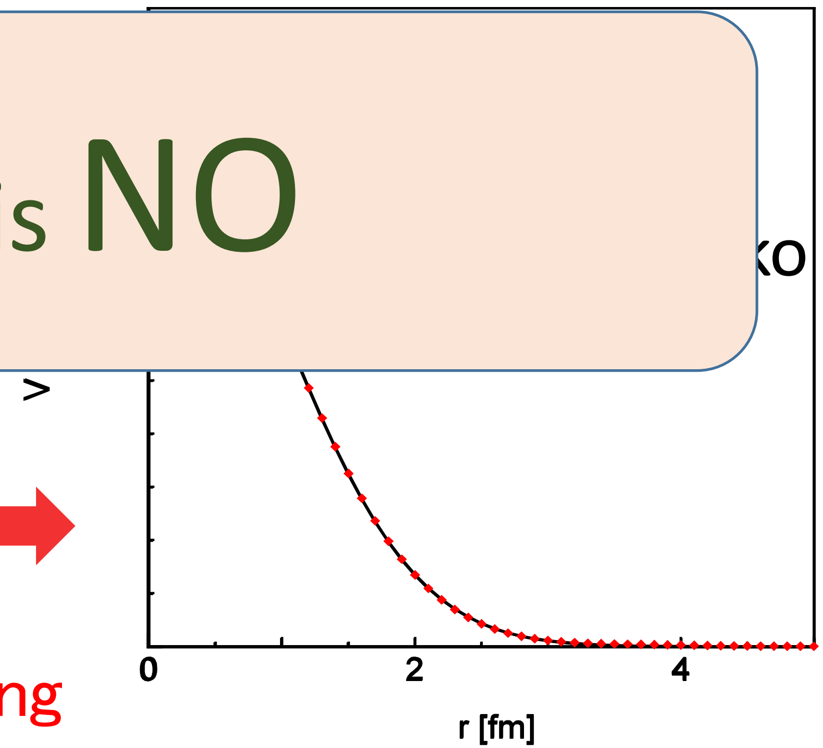
● example

Can this be the end of the story ?



The answer is **NO**

Inverse scattering



2-2. Why NO ? --- current status vs. present theory

Current status

- Formulation in one−dimension via partial wave decomposition
- Assumption of local potential $V(r)$ $\longrightarrow$ excludes nonlocal $V(r,r')$
- Physical meaning is not clear
- (Theory in coordinate space)

Present theory

- Direct formulation in three−dimensional space
  $\longrightarrow$ no assumption of symmetry
- General nonlocal potential $V(r,r')$ is included
- Physical meaning is clear
- (Theory in momentum space)

3. Generalized Optical Theorem (GOT)

3-1. Scattering theory in momentum space

$$H_0|\boldsymbol{k}\rangle = k^2|\boldsymbol{k}\rangle$$
$$H|\boldsymbol{k}\rangle_+ = k^2|\boldsymbol{k}\rangle_+$$

T-matrix

- $\mathcal{T}$: T-operator

$$\mathcal{T}(E) = V + V\frac{1}{E-H+i\eta}V = V + V\frac{1}{E-H_0+i\eta}\mathcal{T}(E)$$

- T : Half-on-shell (HOS) T-matrix $k' \neq k$

$$\langle\boldsymbol{k}'|T|\boldsymbol{k}\rangle = \langle\boldsymbol{k}'|V|\boldsymbol{k}\rangle_+ = \langle\boldsymbol{k}'|\mathcal{T}(k^2)|\boldsymbol{k}\rangle$$

$$\langle\boldsymbol{k}'|T|\boldsymbol{k}\rangle = \langle\boldsymbol{k}'|V|\boldsymbol{k}\rangle + \int\frac{d\boldsymbol{p}}{(2\pi)^3}\langle\boldsymbol{k}'|V|\boldsymbol{p}\rangle\frac{1}{k^2-p^2+i\eta}\langle\boldsymbol{p}|T|\boldsymbol{k}\rangle$$

- t : On-shell (OS) T-matrix $k' \to k$

$$\langle\hat{\boldsymbol{k}}'|t(k)|\hat{\boldsymbol{k}}\rangle = \langle k\hat{\boldsymbol{k}}'|T|k\hat{\boldsymbol{k}}\rangle = \langle k\hat{\boldsymbol{k}}'|\mathcal{T}(k^2)|k\hat{\boldsymbol{k}}\rangle \longrightarrow S(k) = 1 - 2\pi i\rho_k\, t(k)$$

Scattering state

- $$|\boldsymbol{k}\rangle_+ = |\boldsymbol{k}\rangle + \frac{1}{k^2-H_0+i\eta}V|\boldsymbol{k}\rangle_+ = |\boldsymbol{k}\rangle + \frac{1}{k^2-H_0+i\eta}\mathcal{T}(k^2)|\boldsymbol{k}\rangle$$

$$\langle\boldsymbol{p}|\boldsymbol{k}\rangle_+ = (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{k}) + \frac{1}{k^2-p^2+i\eta}\langle\boldsymbol{p}|T|\boldsymbol{k}\rangle$$

3-2. How much is arbitrary in $\langle \boldsymbol{k}' | T | \boldsymbol{k} \rangle$?

HOS T-matrix is closer to observables than pot.

$$\langle k' | V | k \rangle \rightarrow \langle k' | T | k \rangle \rightarrow \text{observables}$$

Optical Theorem (GOT)

- studied only for $\langle k' | T_l | k \rangle$

$$\langle \boldsymbol{k}' | T | \boldsymbol{k} \rangle = \sum_{lm} Y_{lm}(\hat{\boldsymbol{k}}') \langle k' | T_l | k \rangle Y^*_{lm}(\hat{\boldsymbol{k}})$$

M.Baranger et.al., Nucl.Phys.A138(1961)1

- single equation only is known for $\langle \boldsymbol{k}' | T | \boldsymbol{k} \rangle$

Low equation

Its meaning is, however, unknown.

F.E.Low, Phys.Rev.97,1392(1955)
K.Takayanagi, Phys.Rev.A77, 062714(2008)

3-3. Condition for HOS T-matrix (no bound state, for simplicity)

- Hermitian $V(r,r')$ $\Longleftarrow$ Generally nonlocal

$\Updownarrow$

✓ Be careful.
Product of singular factors.

- Completeness of $\{|\boldsymbol{k}\rangle_+\}$

$$\int \frac{d\boldsymbol{p}}{(2\pi)^3}\ \underline{\langle \boldsymbol{k}'|\boldsymbol{p}\rangle_+\ {}_+\langle \boldsymbol{p}|\boldsymbol{k}\rangle} = (2\pi)^3\delta(\boldsymbol{k}'-\boldsymbol{k})$$

$\Downarrow$ $\longleftarrow$

$$\boxed{\langle \boldsymbol{p}|\boldsymbol{k}\rangle_+ = (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{k}) + \frac{1}{k^2-p^2+i\eta}\langle \boldsymbol{p}|T|\boldsymbol{k}\rangle}$$

Ⓐ
$$\boxed{\langle \boldsymbol{k}'|T^\dagger|\boldsymbol{k}\rangle - \langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3}\ \langle \boldsymbol{k}'|T|\boldsymbol{p}\rangle \left(\frac{1}{k'^2-p^2-i\eta} - \frac{1}{k^2-p^2+i\eta}\right)\langle \boldsymbol{p}|T^\dagger|\boldsymbol{k}\rangle}$$

- orthogonality of $\{|\boldsymbol{k}\rangle_+\}$

$$\int \frac{d\boldsymbol{p}}{(2\pi)^3}\ \underline{{}_+\langle \boldsymbol{k}'|\boldsymbol{p}\rangle\langle \boldsymbol{p}|\boldsymbol{k}\rangle_+} = (2\pi)^3\delta(\boldsymbol{k}'-\boldsymbol{k})$$

$\Downarrow$ $\longleftarrow$

$$\boxed{\langle \boldsymbol{p}|\boldsymbol{k}\rangle_+ = (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{k}) + \frac{1}{k^2-p^2+i\eta}\langle \boldsymbol{p}|T|\boldsymbol{k}\rangle}$$

Ⓑ
$$\boxed{\langle \boldsymbol{k}'|T^\dagger|\boldsymbol{k}\rangle - \langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3}\ \langle \boldsymbol{k}'|T^\dagger|\boldsymbol{p}\rangle \left(\frac{1}{k'^2-p^2-i\eta} - \frac{1}{k^2-p^2+i\eta}\right)\langle \boldsymbol{p}|T|\boldsymbol{k}\rangle}$$

3-4. Generalized optical theorem (GOT)

Completeness and orthogonality of scattering states $\{|\boldsymbol{k}\rangle_+\}$

Ⓐ $$\langle \boldsymbol{k}'|T^\dagger|\boldsymbol{k}\rangle - \langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle \boldsymbol{k}'|T|\boldsymbol{p}\rangle \left(\frac{1}{k'^2 - p^2 - i\eta} - \frac{1}{k^2 - p^2 + i\eta} \right) \langle \boldsymbol{p}|T^\dagger|\boldsymbol{k}\rangle$$

- integral over on−shell momentum
- Low equation

Ⓑ $$\langle \boldsymbol{k}'|T^\dagger|\boldsymbol{k}\rangle - \langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle \boldsymbol{k}'|T^\dagger|\boldsymbol{p}\rangle \left(\frac{1}{k'^2 - p^2 - i\eta} - \frac{1}{k^2 - p^2 + i\eta} \right) \langle \boldsymbol{p}|T|\boldsymbol{k}\rangle$$

Ⓐ Ⓑ are independent equations

Optical theorem

$$\boxed{t^\dagger(k) - t(k) = 2\pi i \ \rho_k \ t(k) t^\dagger(k)}$$

$$\boxed{t^\dagger(k) - t(k) = 2\pi i \ \rho_k \ t^\dagger(k) t(k)}$$

$\Updownarrow$

$$S(k)S^\dagger(k) = S^\dagger(k)S(k) = 1$$

T)

- clear physical meaning
  -- orthonormality of $\{|\boldsymbol{k}\rangle_+\}$
- coupled set of nonlinear integral eqs. for HOS T-matrix $\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle$

4. Theory of Inverse scattering

4-1. on-to-one correspondence between V and T

$$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle + \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle \boldsymbol{k}'|V|\boldsymbol{p}\rangle \frac{1}{k^2 - p^2 + i\eta} \langle \boldsymbol{p}|T|\boldsymbol{k}\rangle$$: LS equation

- Prepare Hermitian potential $\langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle$
- Solve LS equation for $\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle$

from V to T

potential
$\langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle$
Hermitian

HOS T-matrix
$\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle$
Generalized
optical theorem Ⓐ Ⓑ

from T to V

- Prepare HOS T-matrix that satisfies Ⓐ Ⓑ
- Solve LS equation for $\langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle$

4-2. inverse scattering problem in momentum space

extension of T from OS to HOS region, satisfying Ⓐ Ⓑ

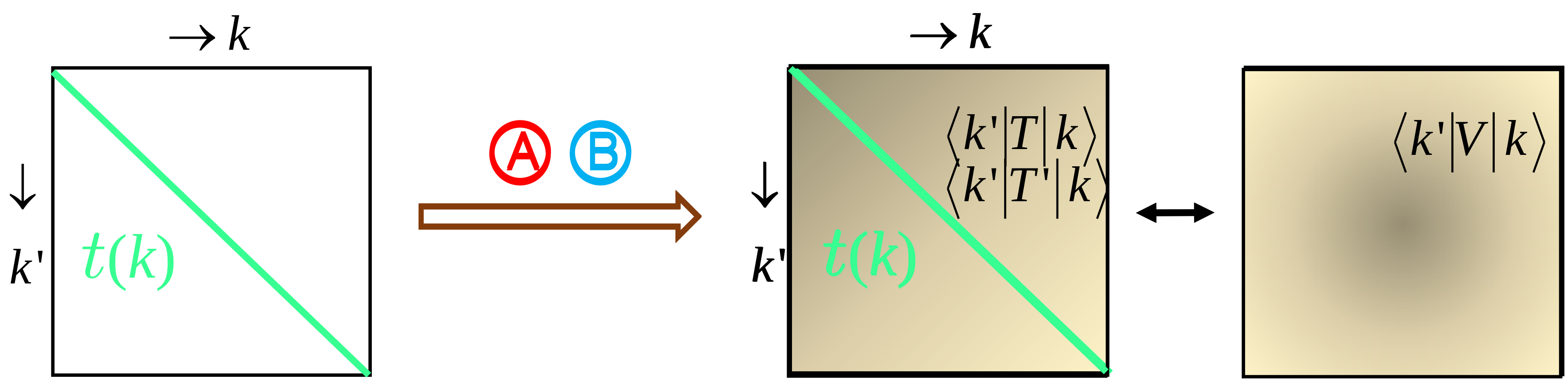
5 variables

6 variables

OS $\langle \hat{\boldsymbol{k}}'|t(k)|\hat{\boldsymbol{k}}\rangle$ $\xRightarrow{Ⓐ\ Ⓑ}$ HOS $\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle$

$\xleftrightarrow[\text{LS}]{} \langle \boldsymbol{k}'|V|\boldsymbol{k}\rangle \longleftrightarrow \langle \boldsymbol{r}'|V|\boldsymbol{r}\rangle$

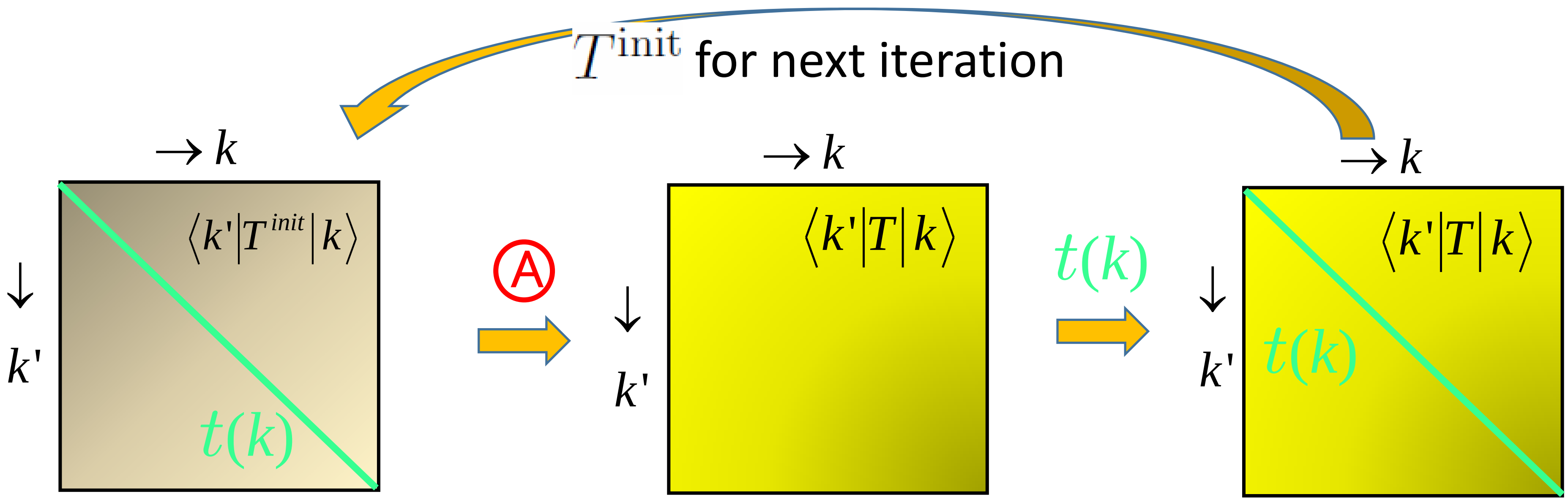
Ⓐ Ⓑ do not uniquely determine $\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle$



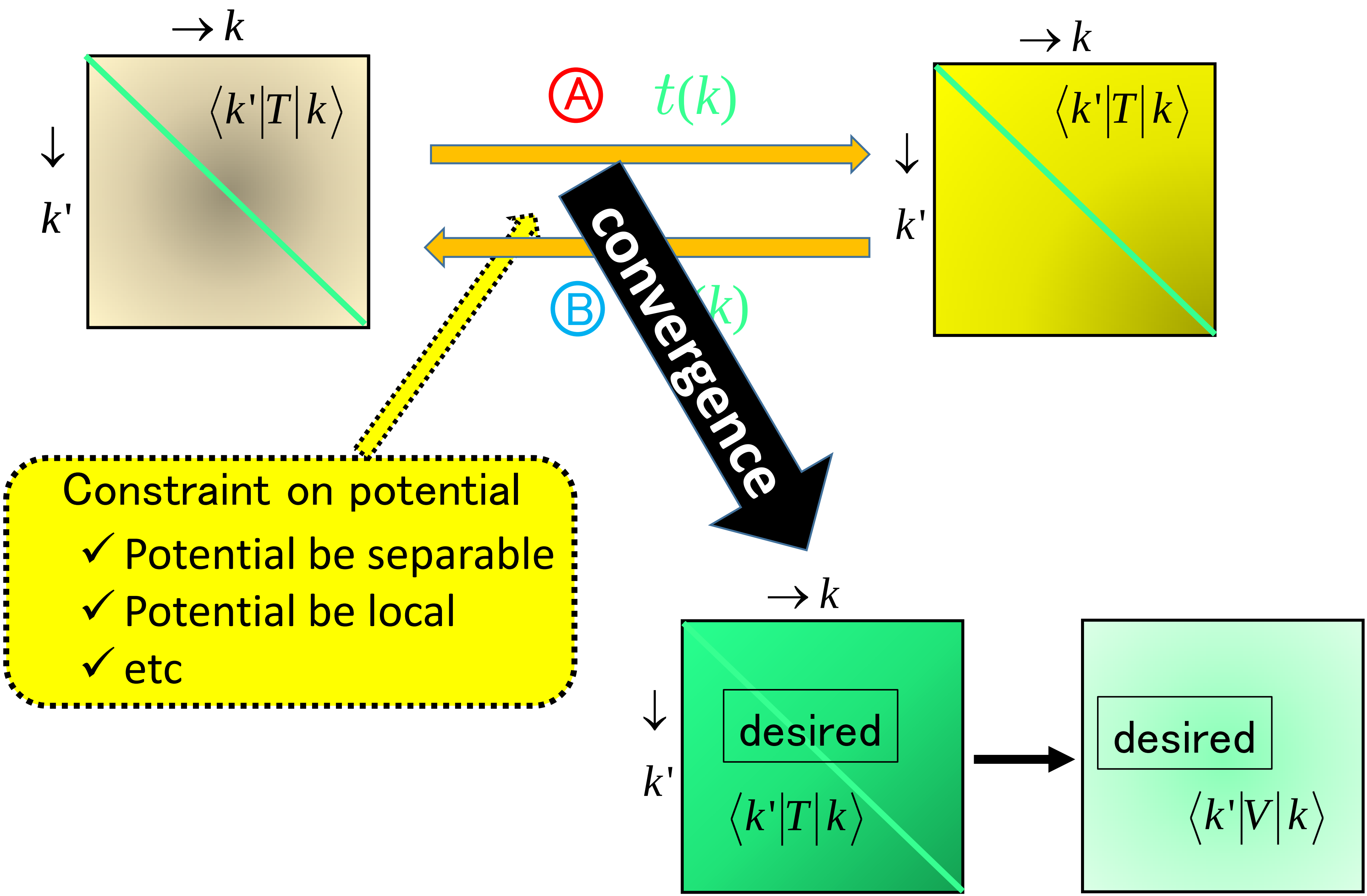
4-3. how to solve Ⓐ (and Ⓑ) for a given $t(k)$

Ⓐ $$\langle \boldsymbol{k}'|\boxed{T^\dagger}|\boldsymbol{k}\rangle - \langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle = \int \frac{d\boldsymbol{p}}{(2\pi)^3} \langle \boldsymbol{k}'|T|\boldsymbol{p}\rangle \left(\frac{1}{k'^2 - p^2 - i\eta} - \frac{1}{k^2 - p^2 + i\eta} \right) \langle \boldsymbol{p}|\boxed{T^\dagger}|\boldsymbol{k}\rangle$$

- nonlinear eq. for T ⇨ iterative solution
- fix $\boxed{T^\dagger}$ via $T^{\rm init}$ ← $\langle k\hat{\boldsymbol{k}}'|T^{\rm init}|k\hat{\boldsymbol{k}}\rangle = \langle \hat{\boldsymbol{k}}'|\underline{t(k)}|\hat{\boldsymbol{k}}\rangle$: OS part fixed
- Ⓐ becomes a linear eq. for T ⇨ solve for T
  → OS part of obtained $\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle$ $\neq$ given $t(k)$
- replace OS part of obtained $\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle$ with given $t(k)$



4-4. solution to inverse scattering problem



5. Test calculation in one-dimension

- S-wave, for simplicity.

$t(k)$ ⟶ $\langle k' | V | k \rangle$

1 variables

2 variables

- Calculation with/without constraint on $\langle k' | V | k \rangle$
    - $t(k)$ of separable potential
    - $t(k)$ of gaussian potential

6. *t(k)* of separable potential

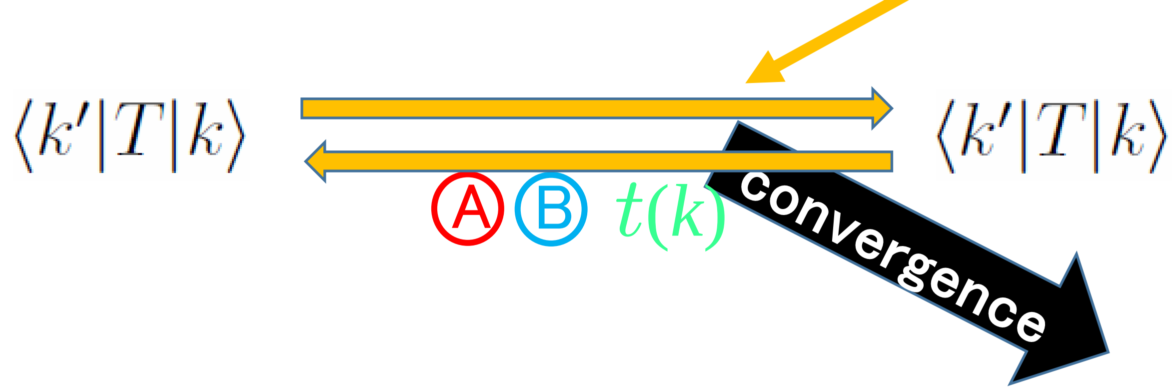
6-1. calculation without constraint

- Separable potential to generate the target $t(k)$

$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2}\frac{4\pi}{1+(\frac{k'}{\beta})^2} \qquad \lambda = -1 \ \ [\mathrm{fm}] \qquad \beta = 1 \ \ [\mathrm{fm}^{-1}]$$

- Calculation without constraint on the solution $\langle k'|V|k\rangle$
  - ▼ $t(k) \longrightarrow \langle k'|V|k\rangle$
  - ▼ Convergent results depend on
    - ➢ initial input $\langle k'|T^{\mathrm{init}}|k\rangle$

$\langle k'|T^{\mathrm{init}}|k\rangle$



● calculation without constraint

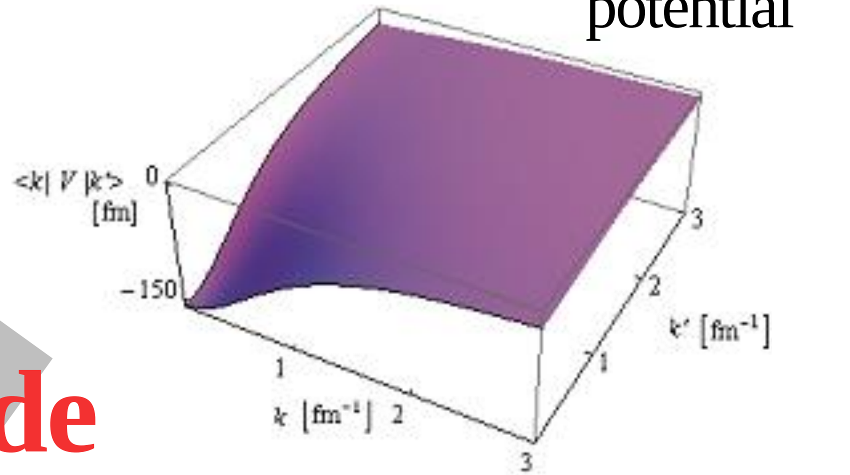
separable potential

$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2}\frac{4\pi}{1+(\frac{k'}{\beta})^2}$$

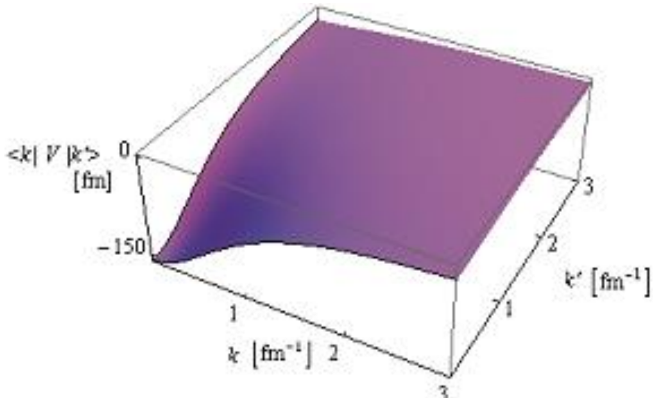
$\lambda = -1$ [fm]

$\beta = 1$ [fm^{-1}]

potential

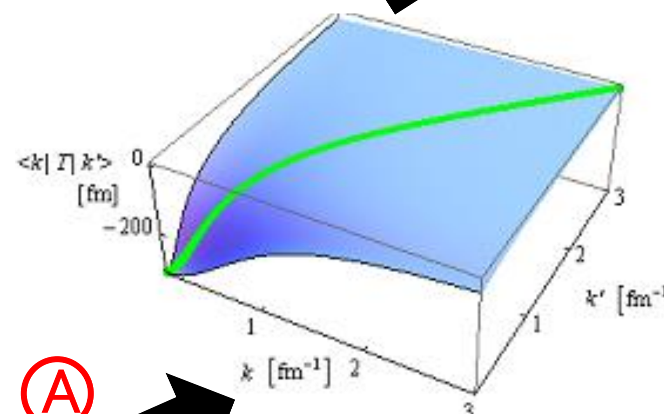


coincide



L-S eq.

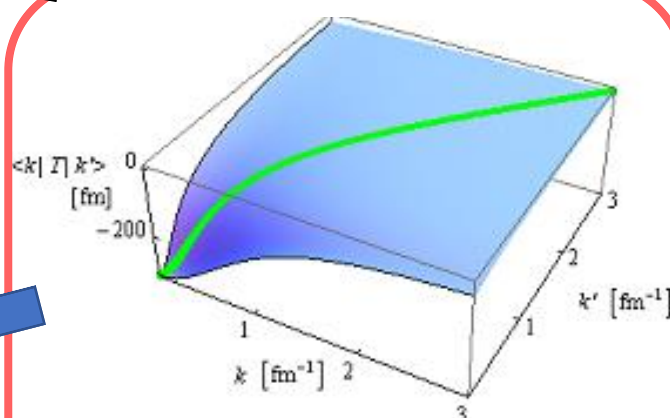
Input



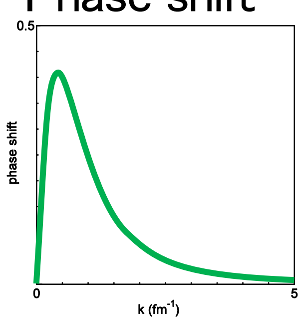
Half-on-Shell T-matrix

Ⓐ

$t(k)$ Ⓑ



Half-on-Shell T-matrix



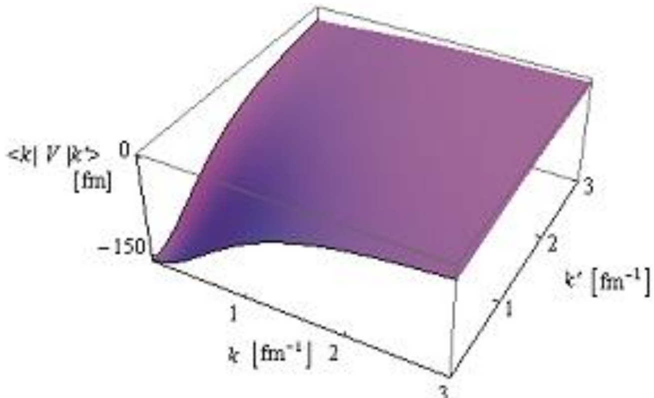
● calculation without constraint

separable potential

$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2}\frac{4\pi}{1+(\frac{k'}{\beta})^2}$$

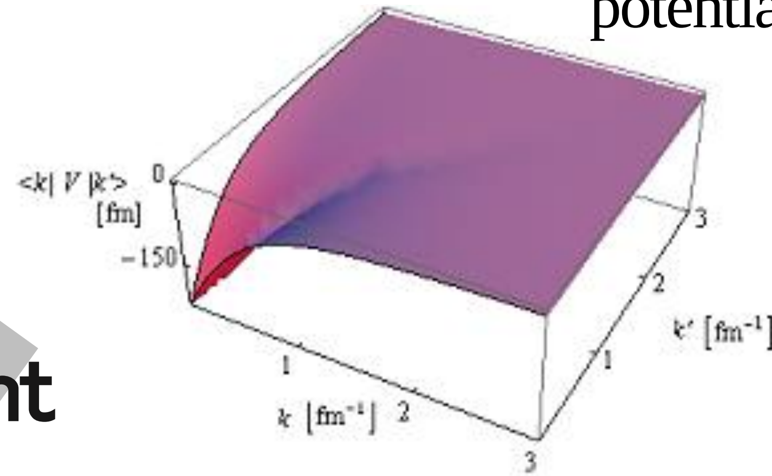
$\lambda = -1$ [fm]

$\beta = 1$ [fm^{-1}]



different

potential

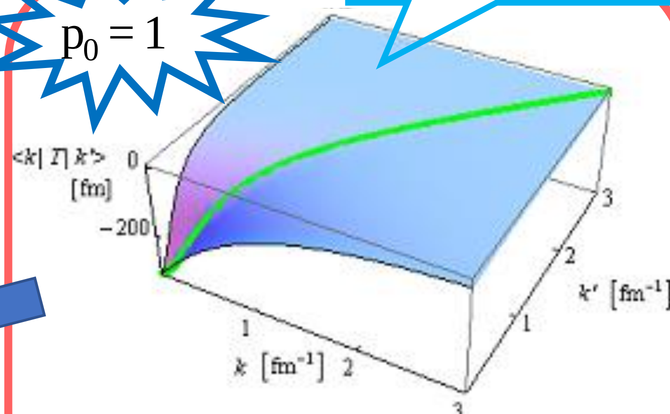


L-S eq.

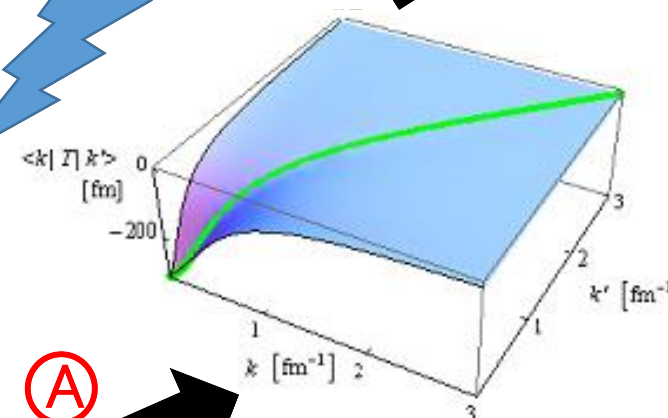
$$\langle k'|T|k\rangle = \langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$

Input

$p_0 = 1$



Half-on-Shell T-matrix

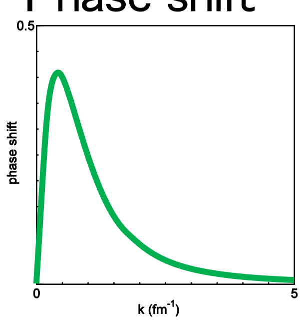


Ⓐ

$t(k)$ Ⓑ

Half-on-Shell T-matrix

Phase shift



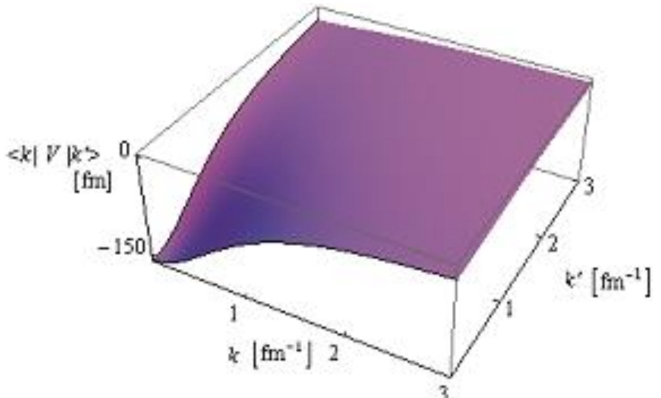
● calculation without constraint

separable potential

$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2}\frac{4\pi}{1+(\frac{k'}{\beta})^2}$$

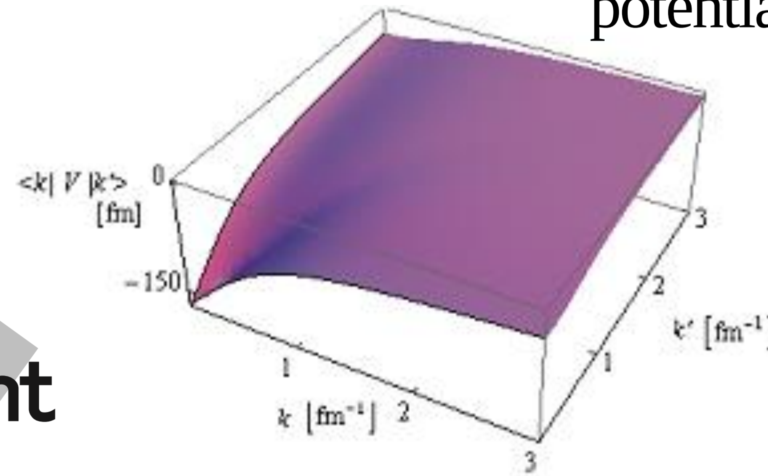
$\lambda = -1$ [fm]

$\beta = 1$ [fm^{-1}]



different

potential

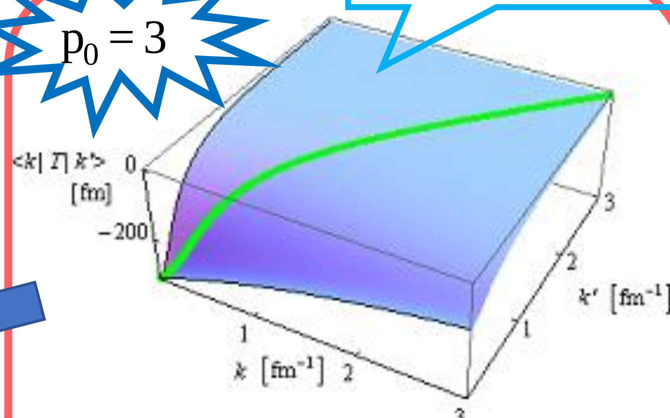


L-S eq.

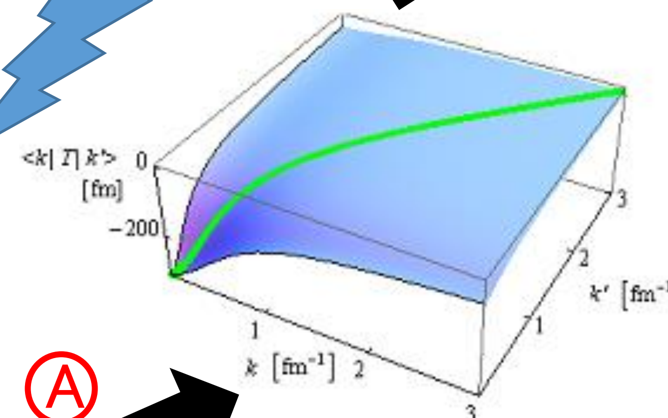
$$\langle k'|T|k\rangle \;\Big\|\; \langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$

Input

$p_0 = 3$



Half-on-Shell T-matrx

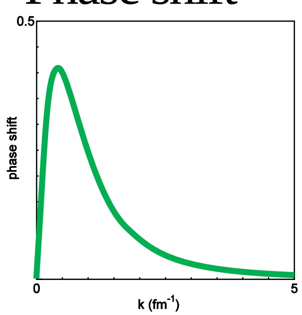


Ⓐ

$t(k)$ Ⓑ

Half-on-Shell T-matrix

Phase shift



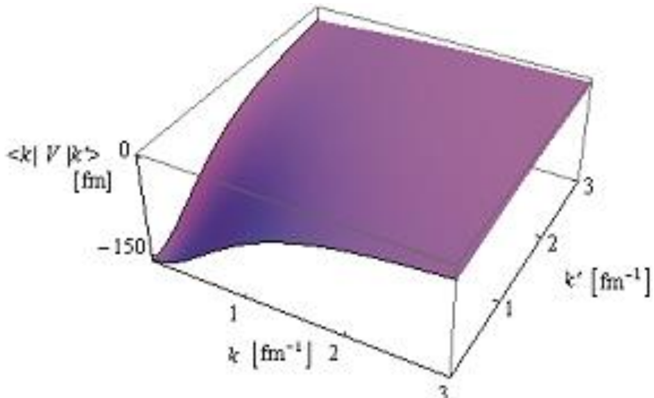
● calculation without constraint

separable potential

$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2}\frac{4\pi}{1+(\frac{k'}{\beta})^2}$$

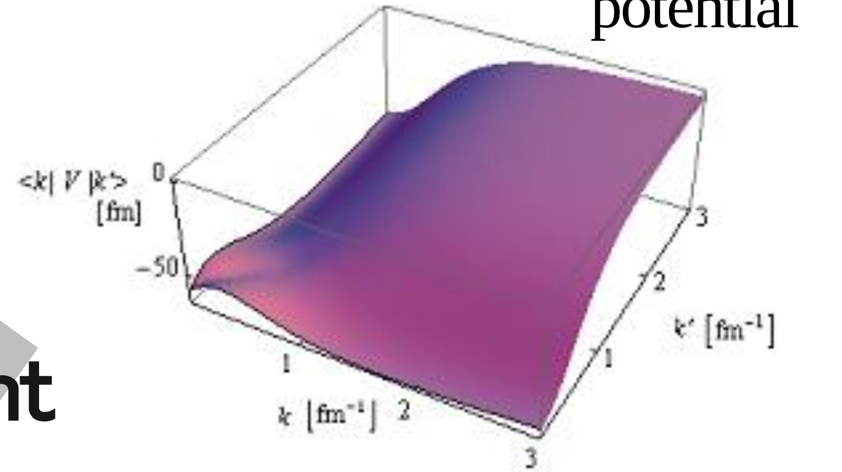
$\lambda = -1$ [fm]

$\beta = 1$ [fm^{-1}]



different

potential

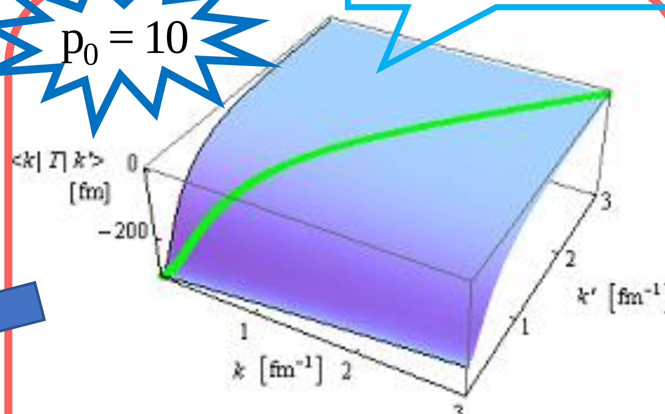


L-S eq.

$$\langle k'|T|k\rangle \;||\; \langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$

Input

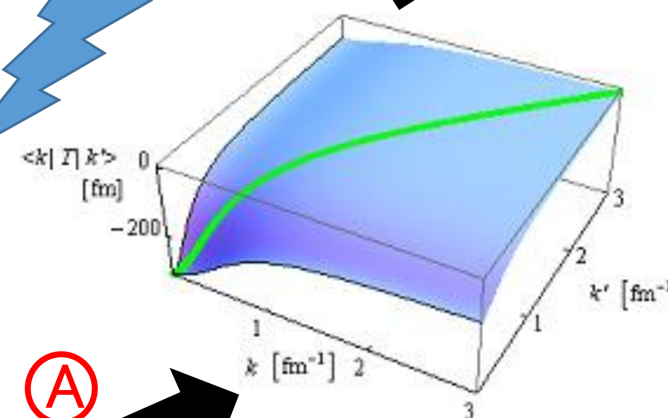
$p_0 = 10$



Half-on-Shell T-matrix

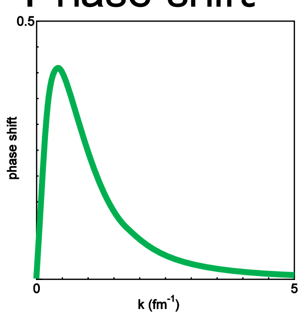
Ⓐ

$t(k)$ Ⓑ



Half-on-Shell T-matrix

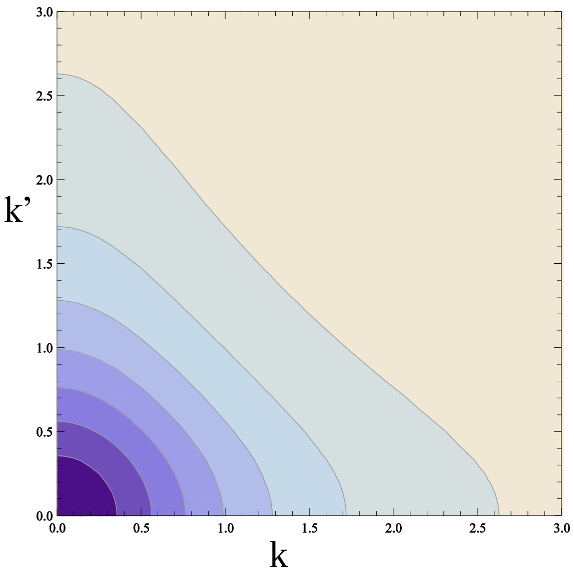
Phase shift

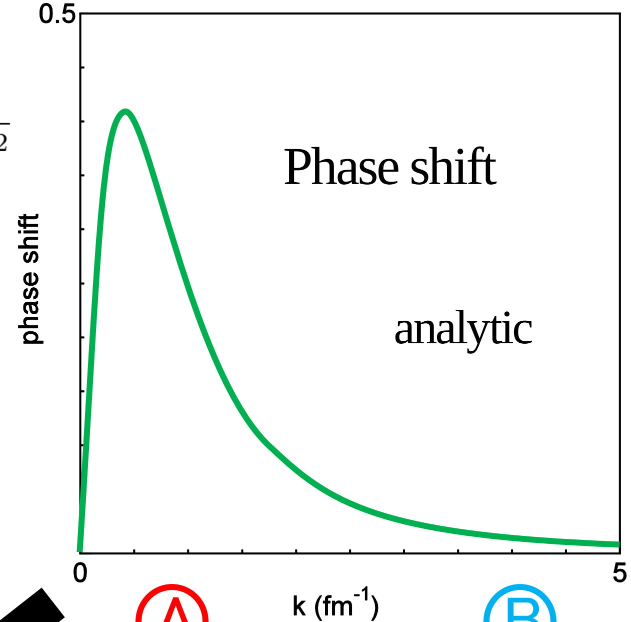


Comparing potentials

separable potential

separable $\langle k|V|k'\rangle = \lambda \dfrac{4\pi}{1+(\frac{k}{\beta})^2}\dfrac{4\pi}{1+(\frac{k'}{\beta})^2}$



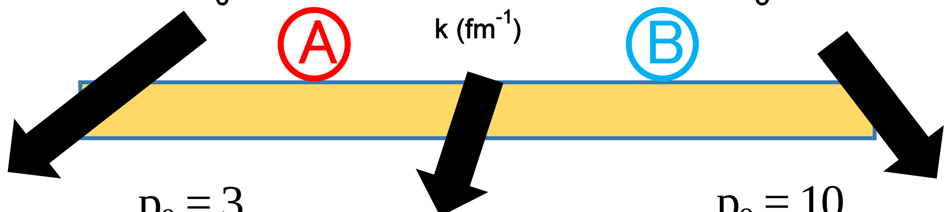


input

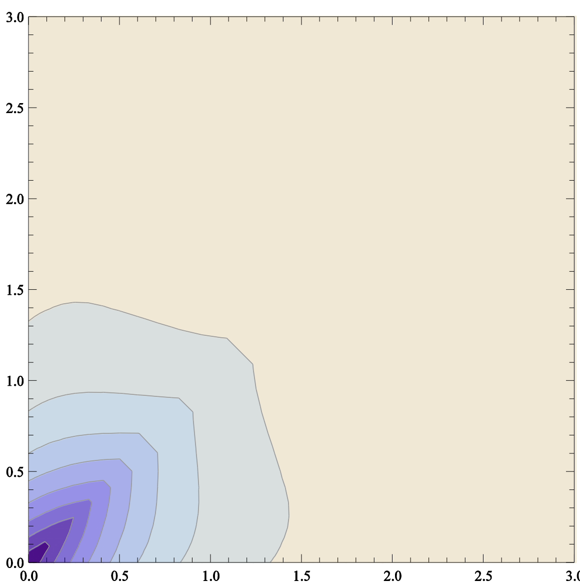
$$\langle k'|T|k\rangle$$

$$||$$

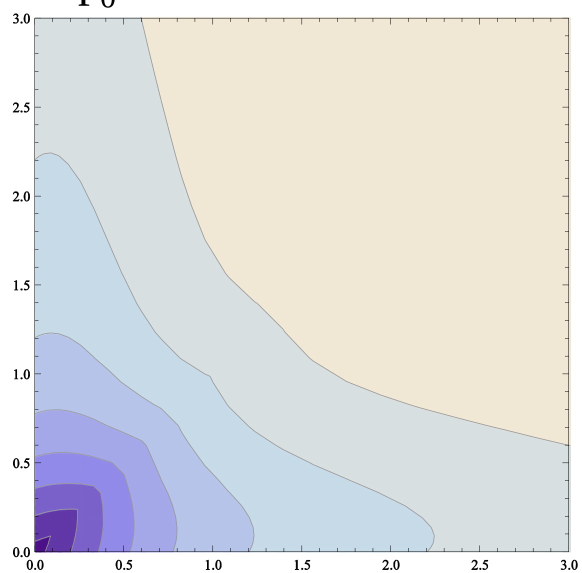
$$\langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$



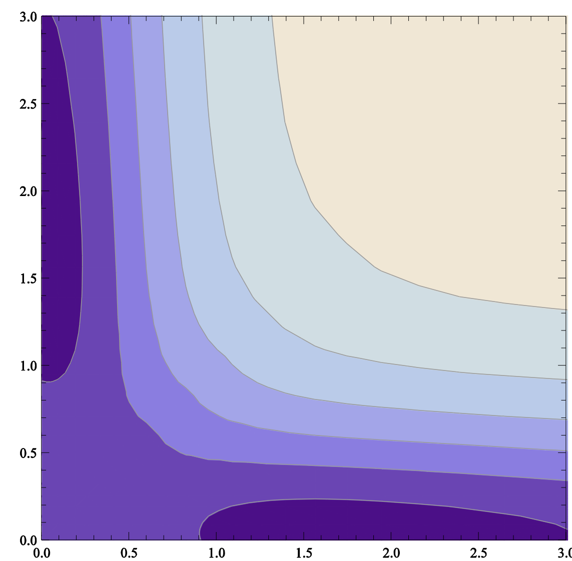
$p_0 = 1$



$p_0 = 3$



$p_0 = 10$



6-2. calculation with constraint

- Separable potential to generate the target $t(k)$

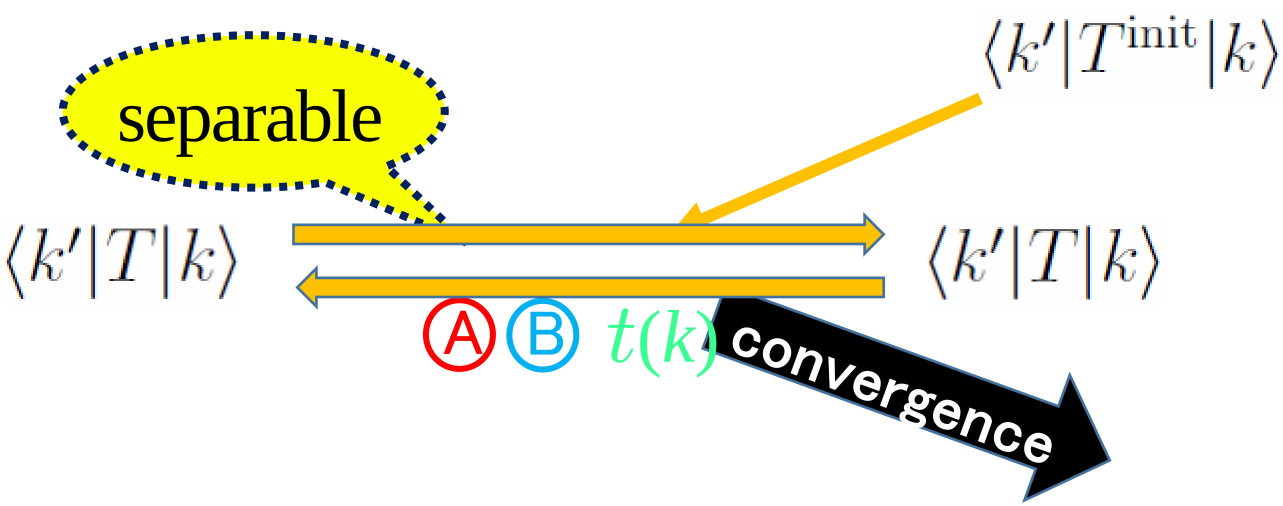
$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2} \frac{4\pi}{1+(\frac{k'}{\beta})^2} \qquad \begin{array}{ll} \lambda = -1 & [\mathrm{fm}] \\ \beta = 1 & [\mathrm{fm}^{-1}] \end{array}$$

- Calculation with constraint

one-to-one

  - $t(k) \longrightarrow \langle k'|V|k\rangle = \lambda g(k')g(k)$ Potential be separable
  - Convergent results do not depend on
    - initial input $\langle k'|T^{\mathrm{init}}|k\rangle$

$\delta(k) \Longleftrightarrow g(k)$



● calculation with constraint

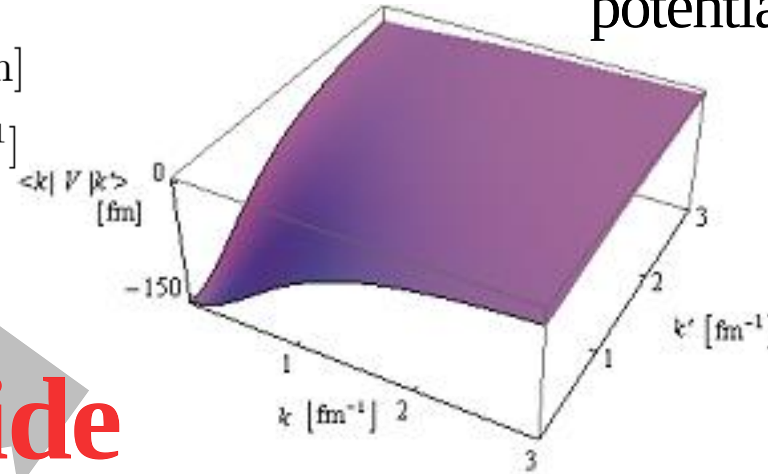
separable potential

$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2}\frac{4\pi}{1+(\frac{k'}{\beta})^2}$$

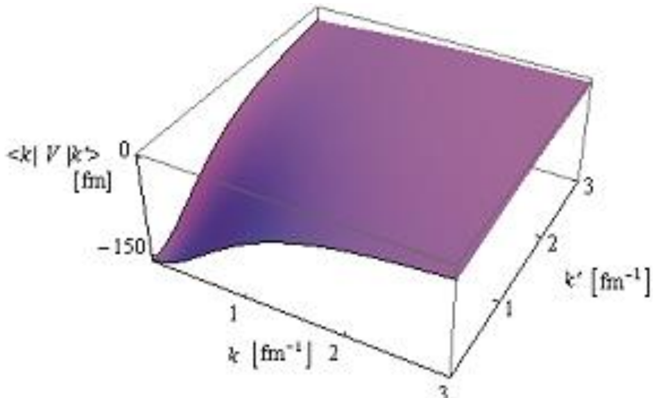
$\lambda = -1$ [fm]

$\beta = 1$ [fm^{-1}]

potential



coincide

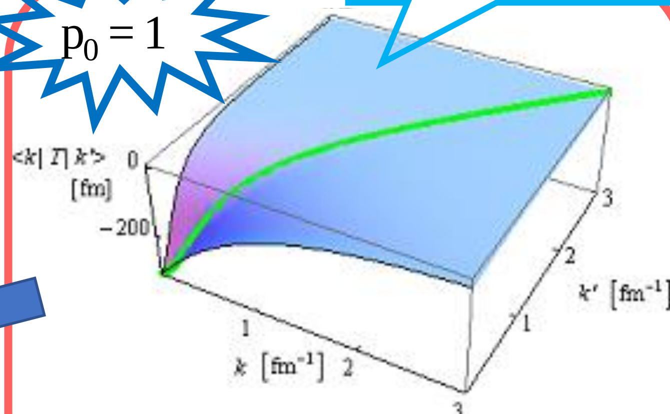


L-S eq.

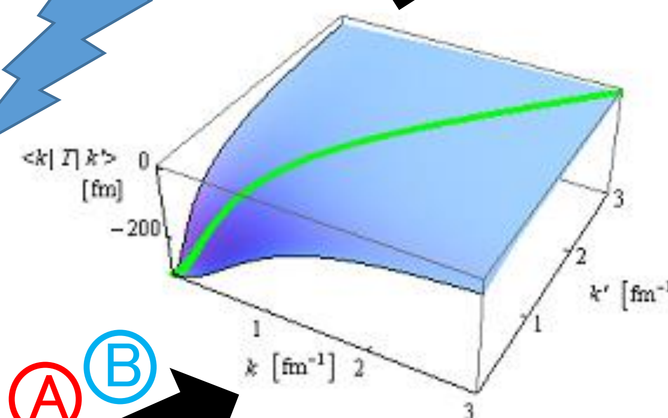
$$\langle k'|T|k\rangle$$
$$||$$
$$\langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$

Input

$p_0 = 1$



Half-on-Shell T-matrix



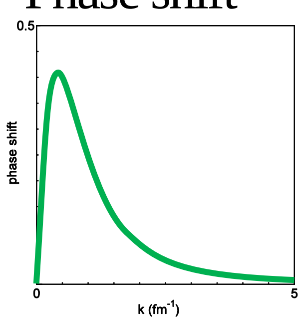
Half-on-Shell T-matrix

Ⓐ Ⓑ

$t(k)$

separable

Phase shift



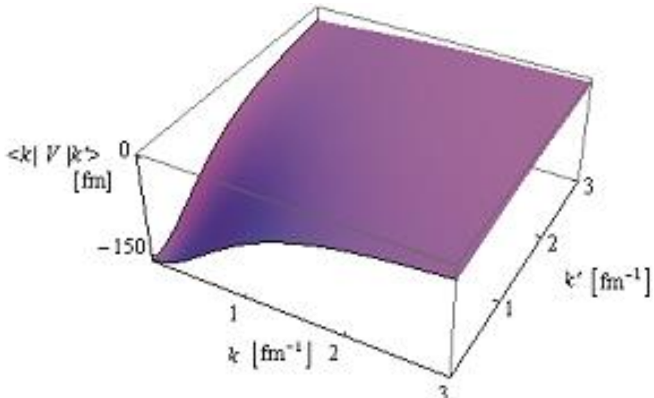
● calculation with constraint

separable potential

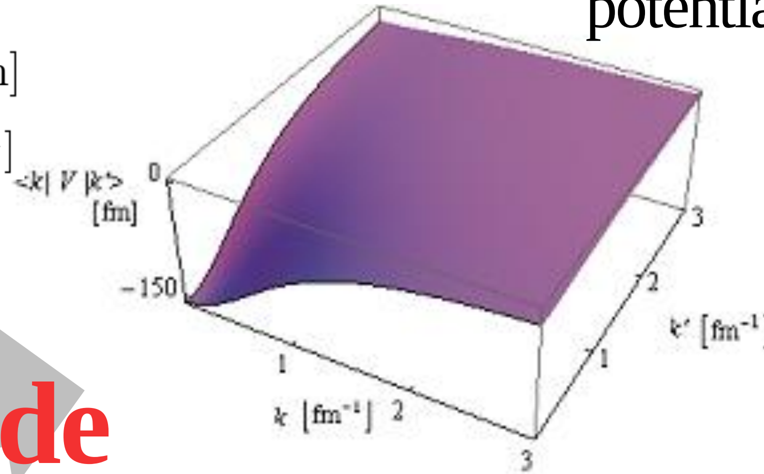
$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2}\frac{4\pi}{1+(\frac{k'}{\beta})^2}$$

$\lambda = -1$ [fm]

$\beta = 1$ [fm^{-1}]



potential



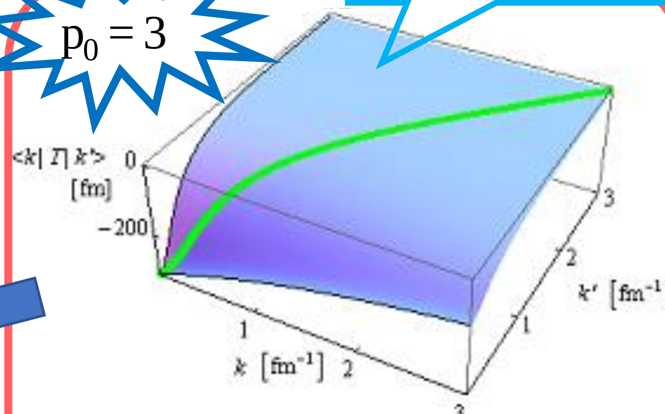
coincide

L-S eq.

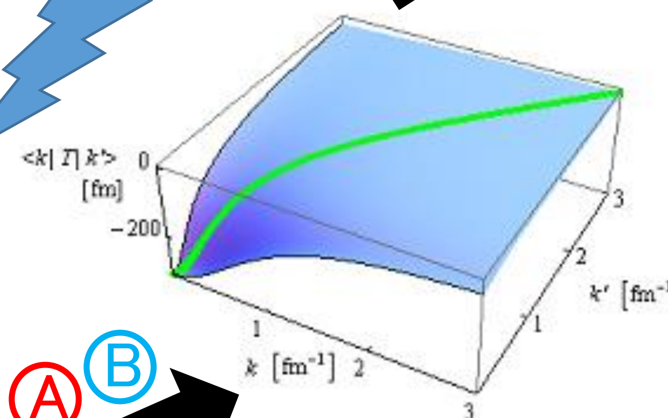
Input

$p_0 = 3$

$$\langle k'|T|k\rangle$$
$$||$$
$$\langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$



Half-on-Shell T-matrix



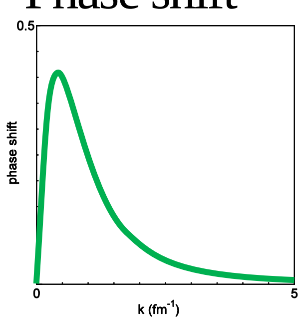
Half-on-Shell T-matrix

Ⓐ Ⓑ

$t(k)$

separable

Phase shift



● calculation with constraint

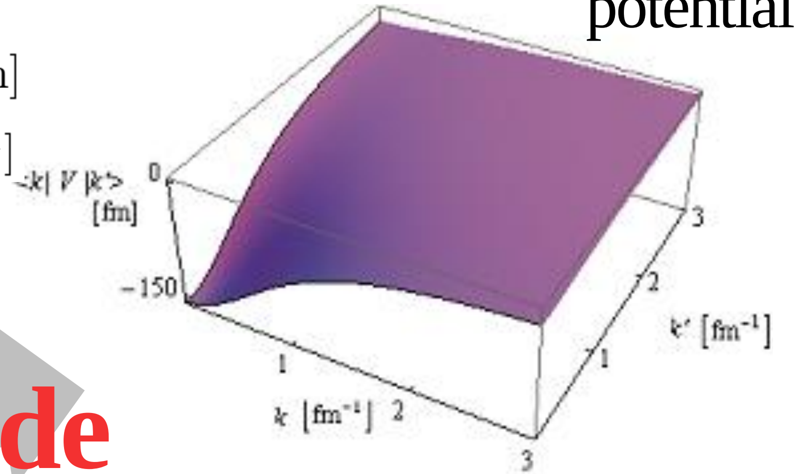
separable potential

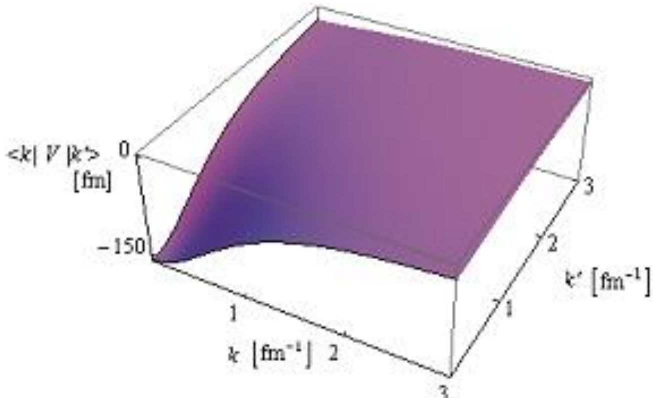
$$\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2} \frac{4\pi}{1+(\frac{k'}{\beta})^2}$$

$\lambda = -1$ [fm]

$\beta = 1$ [fm^{-1}]

potential

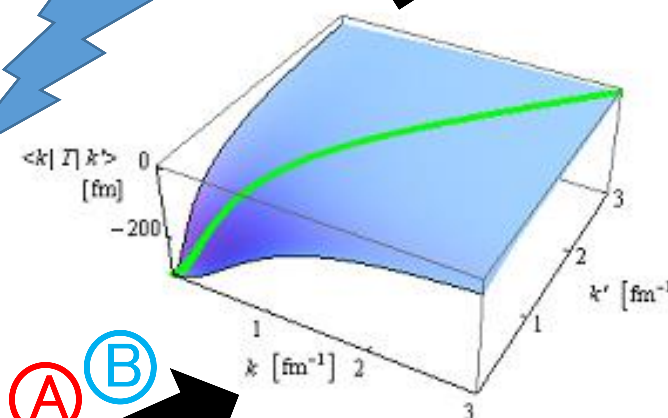




coincide

L-S eq.

$$\langle k'|T|k\rangle$$
$$||$$
$$\langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$



Half-on-Shell T-matrix

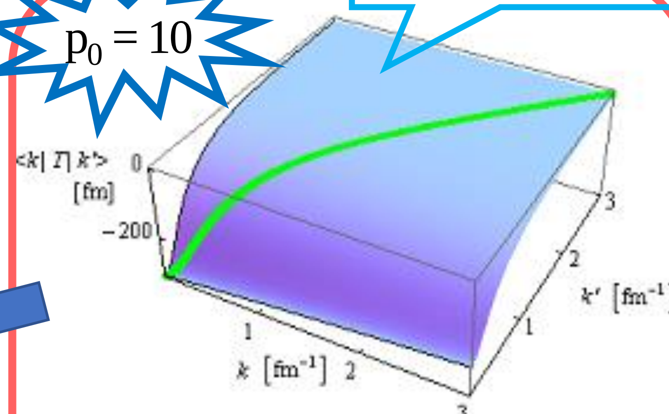
Ⓐ Ⓑ

$t(k)$

separable

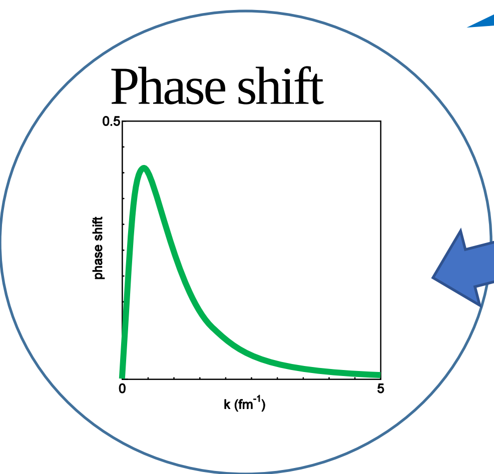
Input

$p_0 = 10$



Half-on-Shell T-matrix

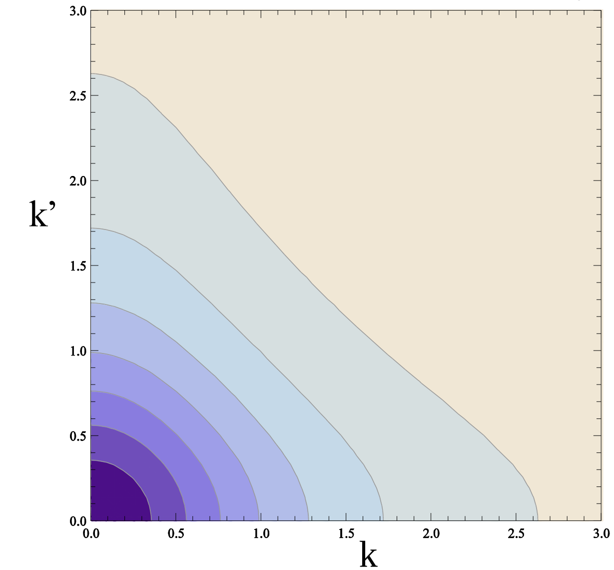
Phase shift

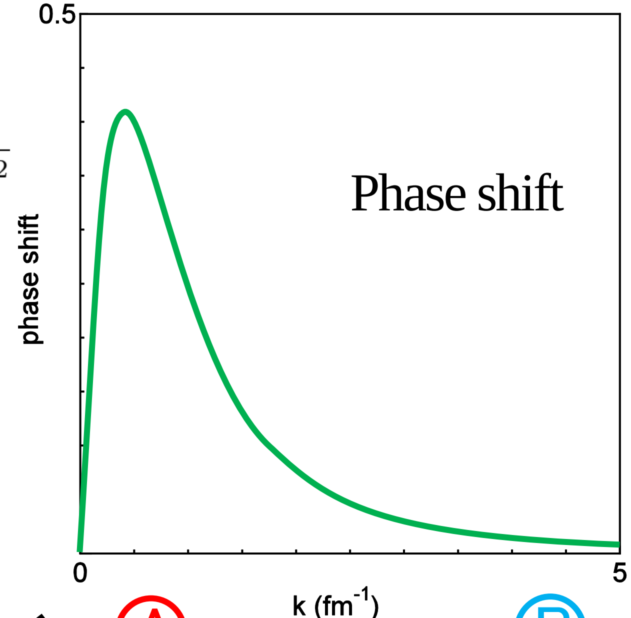


Comparing potentials

separable potential

separable $\langle k|V|k'\rangle = \lambda \frac{4\pi}{1+(\frac{k}{\beta})^2}\frac{4\pi}{1+(\frac{k'}{\beta})^2}$





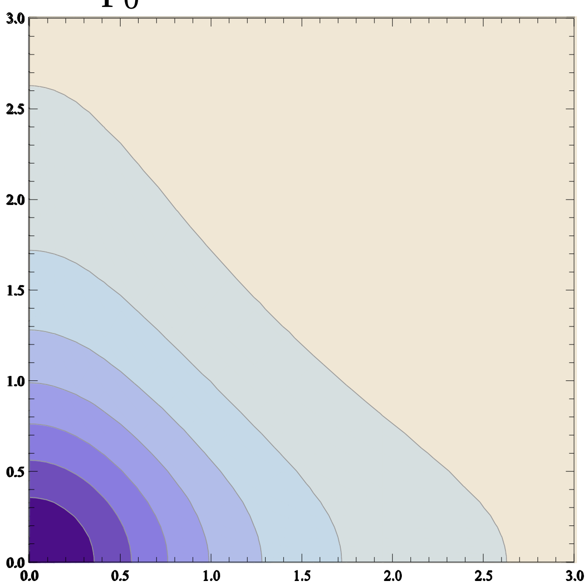
input

$$\langle k'|T|k\rangle$$
$$||$$
$$\langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$

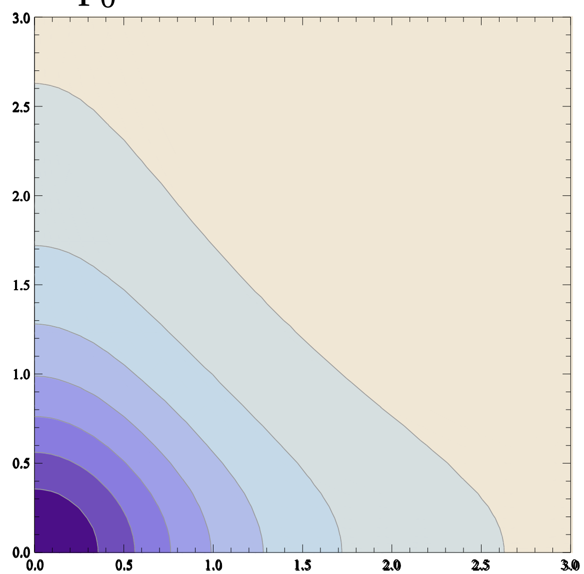
A B

separable

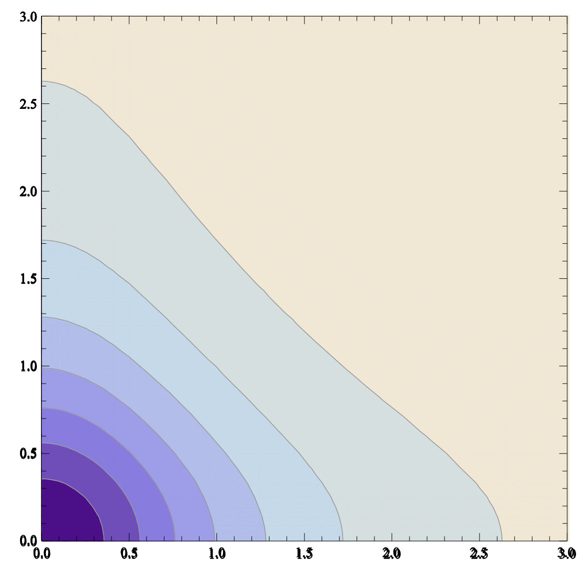
$p_0 = 1$



$p_0 = 3$



$p_0 = 10$



7. *t(k)* of Gaussian potential

- Gaussian potential to generate the target $t(k)$

$$V(r) = V_0 e^{\frac{-r^2}{2b^2}}$$

$$V_0 = 50 \ [\mathrm{MeV}]$$
$$b = 1 \ [\mathrm{fm}]$$

- Calculation without constraint
- Calculation with constraint

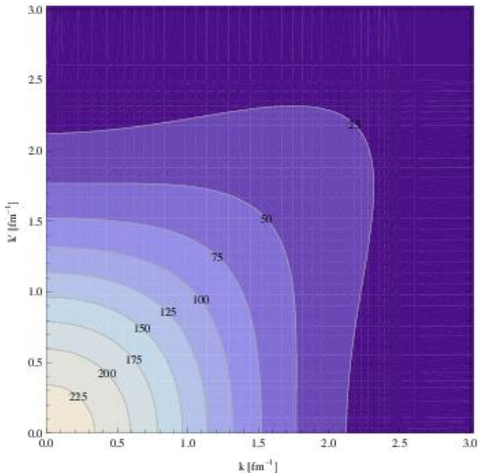
Potential be separable

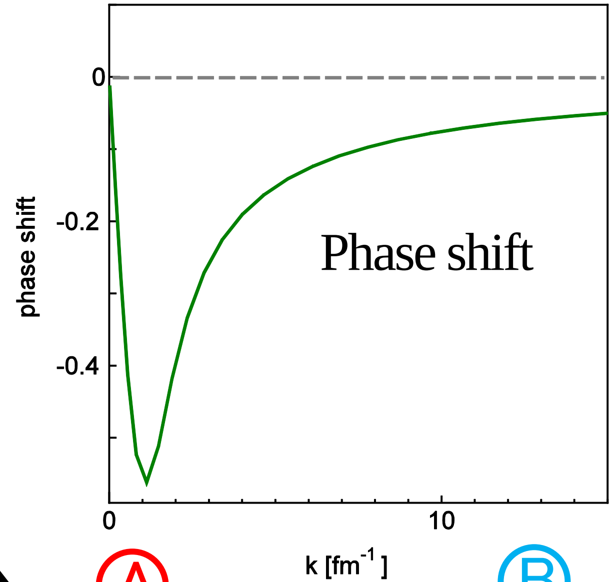
→ Phase shift of a Gaussian potential be reproduced by a separable potential

Comparing potentials

Gaussian potential

exact



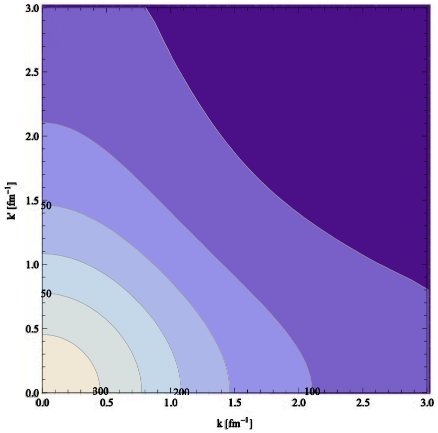


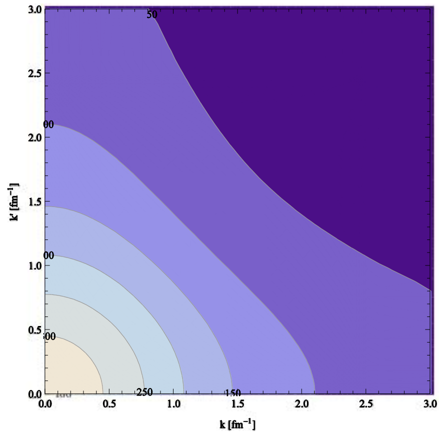
input

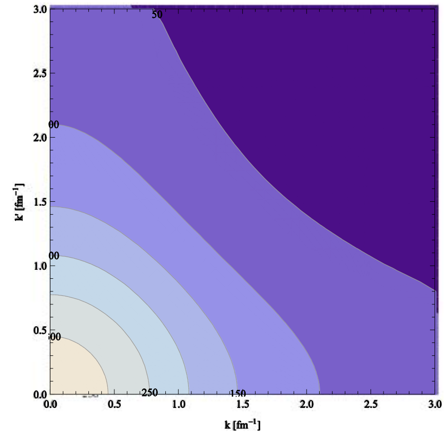
$$\langle k'|T|k\rangle$$
$$||$$
$$\langle k|T|k\rangle\, e^{-\frac{|k'-k|}{p_0}}$$

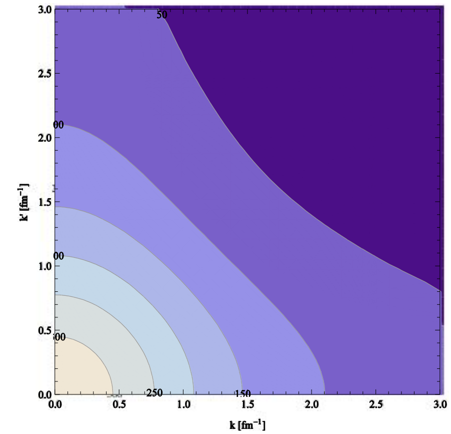
Ⓐ separable Ⓑ

$p_0 = 1$ $\quad p_0 = 3$ $\quad p_0 = 10$









8. summary

- Generalized optical theorem (GOT)
  - Condition for physically acceptable HOS T−matrix Ⓐ Ⓑ
  - Clear physical meaning

    Ⓐ, Ⓑ $\longrightarrow$ completeness, orthogonality of scattering states $\{|\boldsymbol{k}\rangle_+\}$
  - Physics starting from T $\quad \cancel{\langle k|k\rangle} \Longrightarrow \langle k'|T|k\rangle \Longrightarrow$ observable
- Theory of inverse scattering in 3D space
  - Direct solution in 3D space. No symmetry assumption is necessary.
  - Maximum degrees of freedom to reproduce $t(k)$

    solution to Ⓐ Ⓑ with arbitrary constraint

constraint

$\langle \boldsymbol{k'}|T^{\mathrm{init}}|\boldsymbol{k}\rangle$

$\langle k'|T|k\rangle \rightleftarrows \langle k'|T|k\rangle$

Ⓐ Ⓑ $t(k)$

convergence

desired $\langle \boldsymbol{k}'|T|\boldsymbol{k}\rangle \Longrightarrow$ desired $\langle \boldsymbol{r}'|V|\boldsymbol{r}\rangle$

- Successful demonstration

Thank you